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The Quantization of Higher Order Regular Lagrangians as First Order Singular Lagrangians

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In this paper, systems with higher order regular lagrangians are reduced to be first-order singular lagrangians using constrained auxiliary description. The new extended lagrangians are investigated using the Hamilton-Jacobi formulation. Besides, the action function is obtained and the system is quantized using the WKB approximation.

KEY WORDS: higher order lagrangian; singular systems; Hamiltonian formulation; WKB approximation.

1. INTRODUCTION

The Hamiltonian formulation for singular lagrangian has been initiated by Dirac (1950, 1964). This subject has defined an area of specialization in mathematical physics (Faddav and Jackiw, 1988; Henneaux and Teitelboim, 1992; Longhi and Lasanna, 1987). Dirac's approach distinguishes between two types of constraints, the first-and second-class. Many physicists believe that, this destination is important in the classical theories as in quantum theories.

More recently, an approach based on Hamilton-Jacobi formalism was developed to study singular first-order systems (Guler, 1992, 1996; Rabei and Guler, 1992a, 1992b). In this approach, the equations of motions are written as total differential equations in many variables. Besides, there is no need to distinguish between the two types of constraints. In Rabei *et al.* (2002), the action integral is determined and systems with first order Lagrangian are quantized using the WKB approximation. In addition, the Lagrangian with linear velocities are quantized in Muslih *et al.* (2005).

The Hamiltonian formulation for systems with higher order Lagrangian initiated by Ostrogradsky (1850). This formalism seems to be different from the conventional canonical formalism. The structure of phase space and its simplistic

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geometry is not immediately transparent. This leads to confusion when considering canonical or path integral quantization.

The higher-order Lagrangians treated as singular first order Lagrangians in Pons (1989). A canonical formalism presented using the Dirac's method for constrained systems. In this work, systems with regular higher order Lagrangian treated as systems with singular first order Lagrangian using the Hamilton-Jacobi approach (Guler, 1992, 1996; Rabei and Guler, 1992a, 1992b). The action function is determined using the proposed theory given in Nawafleh *et al.* (2004). Besides, this action used to determine the solutions of the equations of motion for higher-order Lagrangians. Then, systems with higher-order Lagrangians quantized using the WKB approximation.

2. REVIEW OF THE HAMILTON-JACOBI FORMULATION

For any physical system, the Lagrangian $L = L(q_i, \dot{q}_i)$, $i = 1, 2, \ldots, N$ is called regular if the rank of the Hessian matrix $W_{ij} = \frac{\partial^2 L}{\partial q_i^\bullet \partial q_j^\bullet}$ is N. Otherwise, it is called singular.

The generalized momenta p_i, corresponding to the generalized coordinates q_i defined as:

$$p_i = \frac{\partial L}{\partial q_i^\bullet}, \qquad i = 1, 2, \ldots, N \tag{2.1}$$

If the rank of the Hessian matrix is $(N - R)$ then, the definition (2.1) leads to relations of the form:

$$H'_\mu(q_i, p_n) = p_\mu + H_\mu = 0, \qquad \mu = N - R + 1, \ldots, N; \tag{2.2}$$

Following to Dirac these relations are called primary constraints (Dirac, 1950, 1964).

Following Guler (1992, 1996) and Rabei and Guler (1992a, 1992b) the corresponding set of the HJPDE's written as:

$$\begin{aligned}
H'_0 &= p_0 + H_0 = \frac{\partial S}{\partial t} + H_0\left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a}\right) = 0\\
H'_\mu &= p_\mu + H_\mu = \frac{\partial S}{\partial q_\mu} + H_\mu\left(q_\beta, q_a, p_a = \frac{\partial S}{\partial q_a}\right) = 0\\
\beta &= 0, N - R + 1, \ldots, N; a = 1, \ldots, N - R
\end{aligned} \tag{2.3}$$

H_0 Being the usual Hamiltonian, and $S(t, q_a, q_\mu)$ being the Hamilton-Jacobi function. The equations of motion written as total differential equations:

$$dq_a = \frac{\partial H_0'}{\partial P_a}dt + \frac{\partial H_\mu'}{\partial p_a}dq_\mu,$$
$$dp_i = -\frac{\partial H_0'}{\partial q_i}dt - \frac{\partial H_\mu'}{\partial q_i}dq_\mu. \tag{2.4}$$

These equations are integrable (Muslih and Guler, 1995; Muslih, 2004) if

$$dH_\mu' = 0 \tag{2.5}$$

These relations are identically satisfied or lead to new secondary constraints. Then one can solve equations (2.4) to obtain the coordinates q_a and momenta p_n as functions of q_μ and t.

A general theory for solving the set of Hamilton-Jacobi partial differential equations for constrained systems (2.3) proposed in Rabei *et al.* (2002). The solution is given in the form:

$$S(t, q_a, q_\mu) = f(t) + W_a(E_a, q_a) + f_\mu(q_\mu) + A \tag{2.6}$$

Where E_a are $(N - R)$ constants of integration, and A is another constant. Here q_μ treated as independent variables, just as the time t, in addition, the equations of motions are obtained using the canonical transformation as follows:

$$\mu_a = \frac{\partial S}{\partial E_a}, \qquad p_i = \frac{\partial S}{\partial q_i}. \tag{2.7}$$

Where, μ_a are constants and they can be determined from the initial conditions.

These equations can be solved to furnish q_a and the momenta p_n as:

$$q_a = q_a(\mu_a, E_a, q_\mu, t), \quad p_i = p_i(\mu_a, E_a, q_\mu, t). \tag{2.8}$$

3. OSTROGRADSKY CONSTRUCTION FOR HIGHER ORDER LAGRANGIAN

The starting point is to consider a lagrangian with N generalized coordinates and depends on up to the m-th time derivatives i.e.

$$L\big(q_i, \dot{q}_i, \ldots, \overset{(m)}{q_i}\big); \qquad \overset{(s)}{q_i} = \frac{d^{(s)} q_i}{dt^{(s)}} \tag{3.1}$$

where $s = 0, 1, \ldots, m$ and $i = 1, \ldots, N$. For such systems the Euler-Lagrange equations of motions, obtained through Hamilton's principle of stationary action, as:

$$\sum_{s=0}^{m}(-1)^s \frac{d^s}{dt^s}\left(\frac{\partial L}{\partial q_i^{(s)}}\right)=0 \tag{3.2}$$

This is a system of N ordinary differential equations of $2m$ − th order, so we need $2mN$ initial conditions to solve it.

The Hamiltonian formalism for theories with higher order derivatives (Ostrogradsky, 1850), treats derivatives $q_i^{(s)}(s=0,\ldots,m-1)$as coordinates. So we will indicate this writing them as $q_i^{(s)}=q_{(s)i}$. In Ostrogradski's formalism, the momenta conjugated respectively to $q_{(m-1)i}$ and $q_{(s-1)i}(s=0,\ldots,m-1)$ introduced as Ostrogradsky (1850).

$$P_{(m-1)i}\equiv\frac{\partial L}{\partial q_i^{(m)}} \tag{3.3}$$

$$P_{(s-1)i}\equiv\frac{\partial L}{\partial q_i^{(s)}}-\dot{p}_{(s)i};\qquad s=1,\ldots,m-1 \tag{3.4}$$

The Hamiltonian defined as:

$$H=\sum_{s=0}^{m-1}P_{(s)i}q_i^{(s+1)}-L\left(q_{i,\ldots,}q_i^{(m)}\right) \tag{3.5}$$

(The Einstein's summation rule for repeated indices has been used throughout this work).

Hamilton's equations of motion are written as

$$\dot{q}_{(s)i}=\frac{\partial H}{\partial q_{(s)i}}=\{q_{(s)i},H\}, \tag{3.6}$$

$$\dot{P}_{(s)i}=-\frac{\partial H}{\partial P_{(s)i}}=\{P_{(s)i},H\}, \tag{3.7}$$

where $\{,\}$ is the Poisson bracket defined as

$$\{A,B\}=\sum_{s=0}^{m-1}\frac{\partial A}{\partial q_{(s)i}}\frac{\partial B}{\partial p_{(s)i}}-\frac{\partial B}{\partial q_{(s)i}}\frac{\partial A}{\partial p_{(s)i}}. \tag{3.8}$$

4. THE HAMILTON-JACOBI TREATMENT OF HIGHER ORDER REGULAR LAGRANGIANS AS FIRST ORDER SINGULAR LAGRANGIANS

For simplicity, let us consider higher order Lagrangian of one degree of freedom,

$$L_0\big(q, \dot{q}, \ldots, q^{(m)}\big) \quad \text{Where} \quad q^{(m)} = \frac{d^m q}{dt^m} \tag{4.1}$$

Now let us introduce new variables q_l and required the constraints.

$$\dot{q}_l = q_{l+1}, \qquad l = 0, 1, \ldots, m-2 \tag{4.2}$$

Where

$$q_0 = q$$

Then, we construct the singular first-order lagrangian as:

$$L_T(q_l, q_{m-1}, \dot{q}_l, \dot{q}_{m-1}, \lambda_l) = L_0(q_l, q_{m-1}, \dot{q}_{m-1}) + \lambda_l(\dot{q}_l - q_{l+1}) \tag{4.3}$$

And the canonical Hamiltonian reads as:

$$H_t(q_l, q_{m-1}, p_{m-1}, \lambda_l) = p_l\dot{q}_l + p_{m-1}\dot{q}_{m-1} + \pi_l\dot{\lambda}_l - L_T(q_l, q_{m-1}, \dot{q}_l, \dot{q}_{m-1}, \lambda_l) \tag{4.4}$$

Where

$$p_{m-1} = \frac{\partial L_T}{\partial \dot{q}_{m-1}}; \quad p_l = \frac{\partial L_T}{\partial \dot{q}_l} = \lambda_l; \quad \pi_l = \frac{\partial L_T}{\partial \dot{\lambda}_l} = 0 \tag{4.5}$$

Following to Guler (1992) and Rabei and Guler (1992), the set of Hamilton-Jacobi partial differential equations takes the form:

$$H_t' = P_t + H_t(q_l, q_{m-1}, p_{m-1}, \lambda_l) \tag{4.6}$$

$$\Phi_l' = \pi_l \tag{4.7}$$

$$H_l' = p_l - \lambda_l \tag{4.8}$$

Where

$$p_t = \frac{\partial S}{\partial t}, \qquad \pi_l = \frac{\partial S}{\partial \lambda_l},$$

$$p_l = \frac{\partial S}{\partial q_l}, \qquad p_{m-1} = \frac{\partial S}{\partial q_{m-1}}.$$

Moreover, the equations of motion can be written as total differential equations as follows:

$$dq_s = \frac{\partial H_t'}{\partial p_s}dt + \frac{\partial \Phi_l'}{\partial p_s}d\lambda_l + \frac{\partial H_l'}{\partial p_s}dq_l \tag{4.9}$$

$$dp_s = -\frac{\partial H_t'}{\partial q_s}dt - \frac{\partial \Phi_l'}{\partial q_s}d\lambda_l - \frac{\partial H_l'}{\partial q_s}dq_l \tag{4.10}$$

$$d\lambda_r = \frac{\partial H_t'}{\partial \pi_r}dt + \frac{\partial \Phi_l'}{\partial \pi_r}d\lambda_s + \frac{\partial H_l'}{\partial \pi_r}dq_l \tag{4.11}$$

$$d\pi_r = -\frac{\partial H_t'}{\partial \lambda_r}dt - \frac{\partial \Phi_l'}{\partial \lambda_r}d\lambda_l - \frac{\partial H_l'}{\partial \lambda_r}dq_l \tag{4.12}$$

where $s = 0, 1, \ldots, m-1$ and $r = 0, 1, \ldots, m-2$ Making use of Eqs. (4.6–8), the previous equations can be written in the following form:

$$dq_l = dq_l \tag{4.13}$$

$$dq_{m-1} = \frac{\partial H_t'}{\partial p_{m-1}}dt \tag{4.14}$$

$$dp_l = -\frac{\partial H_t'}{\partial q_l}dt \tag{4.15}$$

$$dp_{m-1} = -\frac{\partial H_t'}{\partial q_{m-1}}dt \tag{4.16}$$

$$d\lambda_l = d\lambda_l \tag{4.17}$$

$$d\pi_l = -\frac{\partial H_t'}{\partial \lambda_l}dt + dq_l \tag{4.18}$$

The total differential equations are integrable if, and only if,

$$dH_t' = dp_t - dH_t = 0 \tag{4.19}$$

$$dH_l' = dp_l - d\lambda_l = 0 \tag{4.20}$$

$$d\Phi_l' = d\pi_l = 0 \tag{4.21}$$

Using Eq. (4.15), then Eq. (4.20) can be written as:

$$\frac{\partial H_t'}{\partial q_l}dt + d\lambda_l = 0 \tag{4.22}$$

Thus, the integrability conditions lead to:

$$\dot{\lambda}_l = -\frac{\partial H_t'}{\partial q_l} \tag{4.23}$$

In addition, Eqs. (4.18) take the form:

$$\dot{\pi}_l = -\frac{\partial H_t'}{\partial \lambda_l} + \dot{q}_l \tag{4.24}$$

Thus, the equations of motion can be written as:

$$\dot{p}_s = -\frac{\partial H_t'}{\partial q_s} \tag{4.25}$$

$$\dot{q}_r = \frac{\partial H_t'}{\partial \lambda_r} \tag{4.26}$$

$$\dot{q}_{m-1} = \frac{\partial H_t'}{\partial p_{m-1}} \tag{4.27}$$

These equations are equivalent to the following Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}_l}\right) - \frac{\partial L_T}{\partial q_l} = 0 \tag{4.28}$$

$$\frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}_{m-1}}\right) - \frac{\partial L_T}{\partial q_{m-1}} = 0 \tag{4.29}$$

$$\frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{\lambda}_l}\right) - \frac{\partial L_T}{\partial \lambda_l} = 0 \tag{4.30}$$

Equations (4.30) give the constraints (4.2). While, Eqs. (4.28) for the variables q_{l+1} can be written as:

$$\frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}_{l+1}}\right) - \frac{\partial L_T}{\partial q_{l+1}} = 0 \tag{4.31}$$

Making using of Eqs. (4.3) and (4.5), Eqs. (4.31) take the form:

$$\frac{d}{dt}(p_{l+1}) - \frac{\partial L_0}{\partial q_{l+1}} + \lambda_l = 0 \tag{4.32}$$

which, can be written as:

$$p_l = \frac{\partial L_0}{\partial q_{l+1}} - \dot{p}_{l+1} \qquad l = 1, 2, \ldots, m-2 \tag{4.33}$$

These equations can be arranged as:

$$\begin{aligned} p_{l+1} &= \tfrac{\partial L_0}{\partial q_{l+2}} - \dot{p}_{l+2} \\ &\vdots \\ p_{m-2} &= \tfrac{\partial L_0}{\partial q_{m-1}} - \dot{p}_{m-1} \end{aligned} \tag{4.34}$$

Using back substitution, we get

$$\begin{aligned} p_l = \frac{\partial L_0}{\partial q_{l+1}} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial q_{l+2}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial q_{l+3}}\right) + \cdots \\ + (-1)^{m-1}\frac{d^{m-l-1}}{dt^{m-l-1}}\left(\frac{\partial L_0}{\partial \dot{q}_{m-1}}\right). \end{aligned} \tag{4.35}$$

Moreover, these equations represent the momenta in Ostrogradsky construction (Ostrogradsky, 1850).

If $l = 0$, Eq. (4.35) takes the form:

$$p_0 = \frac{\partial L_0}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial q_2}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial q_3}\right) + \cdots + (-1)^{m-1}\frac{d^{m-1}}{dt^{m-1}}\left(\frac{\partial L_0}{\partial \dot{q}_{m-1}}\right). \tag{4.36}$$

Taking the first derivative with respect to t for Eq. (4.36), we get

$$\frac{\partial L_0}{\partial q_0} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial q_1}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial q_2}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L_0}{\partial q_3}\right) + \cdots + (-1)^m\frac{d^m}{dt^m}\left(\frac{\partial L_0}{\partial \dot{q}_{m-1}}\right) = 0. \tag{4.37}$$

This equation can be written as

$$\frac{\partial L_0}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_0}{\partial \dot{q}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L_0}{\partial \ddot{q}}\right) - \frac{d^3}{dt^3}\left(\frac{\partial L_0}{\partial q^{(3)}}\right) + \cdots + (-1)^m\frac{d^m}{dt^m}\left(\frac{\partial L_0}{\partial q^{(m)}}\right) = 0 \tag{4.38}$$

Which can be finally has the form:

$$\sum_{s=0}^{m}(-1)^s\frac{d^{(s)}}{dt^{(s)}}\left(\frac{\partial L_0}{\partial q^{(s)}}\right) = 0. \tag{4.39}$$

This equation is the Euler equation for regular higher order lagrangian (Pimentel and Teixeira, 1998).

4.1. The Hamilton-Jacobi Function

The set of HJPDE's (4.6–8) can be written in the following form:

$$H_t' = \frac{\partial S}{\partial t} + H_t\left(q_l, q_{m-1}, \lambda_l, \frac{\partial S}{\partial q_l}, \frac{\partial S}{\partial q_{m-1}}\right) = 0 \tag{4.40}$$

$$\Phi_l' = \frac{\partial S}{\partial \lambda_l} = 0 \tag{4.41}$$

$$H_l' = \frac{\partial S}{\partial q_l} - \lambda_l = 0 \tag{4.42}$$

Making use of Eq. (2.6) the action function takes the form

$$S(q_{m-1}, q_l, \lambda_l, t) = f(t) + W_{m-1}(E_{m-1}, q_{m-1}) + f_l(q_l) + f_l'(\lambda_l) + A \quad (4.43)$$

Where, E_{m-1} is the constant of integration. Here q_l and λ_l are treated as independent variables as well as the time. The equations of motion are obtained using the canonical transformations (Nawafleh *et al.*, 2004) as follows:

$$\mu_{m-1} = \frac{\partial S}{\partial E_{m-1}}; \quad p_l = \frac{\partial S}{\partial q_l}; \quad \pi_l = \frac{\partial S}{\partial \lambda_l}. \quad (4.44)$$

Where, μ_{m-1} is a constant and can be determined from the initial conditions.

Eqs. (4.44) can be solved to obtain q_{m-1} and the momenta p_l as:

$$q_{m-1} = q_{m-1}(\mu_{m-1}, E_{m-1}, q_l, \lambda_l, t), \quad p_l = p_l(\mu_{m-1}, E_{m-1}, q_l, \lambda_l, t) \quad (4.45)$$

Using Eqs. (4.41) and (4.43) one can find that:

$$\frac{\partial f_l'(\lambda_l)}{\partial \lambda_l} = 0$$

Thus,

$$f_l'(\lambda_l) = \text{constant} \quad (4.46)$$

In addition, from Eq. (4.44), one finds,

$$\frac{\partial f_l(q_l)}{\partial q_l} - \lambda_l = 0$$

$$f_l(q_l) = \lambda_l q_l \quad (4.47)$$

Moreover, using the fact that $f(t) = -E_{m-1}t$ one can write the Hamilton-Jacobi function in the following form:

$$S(q_{m-1}, q_l, \lambda_l, t) = -E_{m-1}t + W_{m-1}(E_{m-1}, q_{m-1}) + \lambda_l q_l + A' \quad (4.48)$$

Where A' is constant.

4.2. The WKB Approximation

A general theory for using the WKB approximation for constrained systems to find the wave function and the connection between the classical and Quantum-mechanical equations of motion has been given in Rabei *et al.* (2002). According to this theory, the wave function for our system can be written as:

$$\psi(q_l, q_{m-1}, \lambda_l, t) = \psi_{0m-1}(q_{m-1}) e^{\frac{iS(q_l, q_{m-1}, \lambda_l, t)}{\hbar}} \quad (4.49)$$

Where

$$\psi_{0m-1}(q_{m-1}) = \frac{1}{\sqrt{p_{m-1}(q_{m-1})}}$$

The wave function in Eq. (4.49) satisfies the following conditions

$$H_t'\psi = p_t\psi + H_t\psi = 0$$
$$\Phi_l'\psi = \pi_l\psi = 0 \tag{4.50}$$
$$H_l'\psi = p_l\psi - \lambda_l\psi = 0$$

These conditions are obtained when the dynamical coordinates and momenta are turned into there corresponding operators:

$$p_s \rightarrow \hat{p}_s = \frac{\hbar}{i}\frac{\partial}{\partial q_s}, \tag{4.51}$$

$$p_0 \rightarrow \hat{p}_0 = \frac{\hbar}{i}\frac{\partial}{\partial t}, \tag{4.52}$$

$$\pi_l \rightarrow \hat{\pi}_l = \frac{\hbar}{i}\frac{\partial}{\partial \lambda_l}, \tag{4.53}$$

5. ILLUSTRATIVE EXAMPLE

Consider the following second-order Lagrangian

$$L = \frac{1}{2}(\ddot{q}^2 - \dot{q}^2) \tag{5.1}$$

Which is describes the one-dimensional motion of black box in which a harmonic oscillator is hidden (a system of units is chosen such that the angular frequency of oscillations is one) (Olga, 1997).

This example has been solved using the Ostrogradsky theory and the results found to be:

$$q = at + b\cos(t + \delta) + c \tag{5.2}$$

$$p_0 = \frac{\partial L}{\partial q^{(1)}} - \dot{p}_1 = -a \tag{5.3}$$

$$p_1 = \frac{\partial L}{\partial q^{(2)}} = -b\cos(t + \delta) \tag{5.4}$$

Where a, b, c and δ are constants.

According to our treatment, the above Lagrangian can be written as:

$$L = \frac{1}{2}\left(\dot{q}_1^2 - q_1^2\right) \quad \text{Where} \quad \dot{q} = \dot{q} \quad 0 = q_1$$

With the aid of Eq. (4.3) the extended lagrangian is

$$L_T = \frac{1}{2}\dot{q}_1^2 - \frac{1}{2}q_1^2 + \lambda_0(\dot{q}_0 - q_1) \tag{5.5}$$

Moreover, using Eq. (4.5), one finds that:

$$p_1 = \frac{\partial L_T}{\partial \dot{q}_1} = \dot{q}_1;$$

$$p_0 = \frac{\partial L_T}{\partial \dot{q}_0} = \lambda_0;$$

$$\pi_0 = \frac{\partial L_T}{\partial \dot{\lambda}_0} = 0.$$

The canonical Hamiltonian can be obtained as:

$$H_t = \frac{p_1^2}{2} + \frac{1}{2}q_1^2 + \lambda_0 q_1 \tag{5.6}$$

Thus, the set of HJPDE's can be written as:

$$\begin{aligned} H'_t &= p_t + H_t = p_t + \frac{p_1^2}{2} + \frac{1}{2}q_1^2 + \lambda_0 q_1 \\ \Phi'_0 &= \pi_0 = 0 \\ H'_0 &= p_0 - \lambda_0 = 0 \end{aligned} \tag{5.7}$$

Using the Eqs. (4.23), (4.25), (4.26) and (4.27), one gets

$$\dot{p}_0 = 0 \tag{5.8}$$

$$\dot{q}_0 = q_1 \tag{5.9}$$

$$\dot{p}_1 = -q_1 - \lambda_0 \tag{5.10}$$

$$q_1^{\bullet} = p_2 \tag{5.11}$$

Equation (5.9) represents the constraint and using the integrability condition (4.23), one finds

$$\dot{\lambda}_0 = -\frac{\partial H'_t}{\partial q_0} = 0 \tag{5.12}$$

Thus, Eq. (5.10) leads to:

$$\ddot{p}_1 + \dot{q}_1 = 0 \tag{5.13}$$

Using Eq. (5.11), one finds:

$$q_1^{(3)} + \dot{q}_1 = 0 \tag{5.14}$$

Where p_2 equals to $\dot{p}_1$, and using Eq. (5.9), we have:

$$q^{(4)} + \ddot{q} = 0 \tag{5.15}$$

Thus, the general solutions are

$$q = at + b\cos(t+\delta) + c \tag{5.16}$$

$$p_0 = \frac{\partial L}{\partial q^{(1)}} - \dot{p}_1 = -a \tag{5.17}$$

$$p_1 = \frac{\partial L}{\partial q^{(2)}} = -b\cos(t+\delta) \tag{5.18}$$

Where, a, b, c are constants. These solutions are in exact agreement with the solutions (5.2–4).

The corresponding HJPDE's are calculated as:

$$H'_t = \frac{\partial S}{\partial t} + \frac{1}{2}\left(\frac{\partial S}{\partial q_1}\right)^2 + \frac{1}{2}q_1^2 + \lambda_0 q_1 = 0 \tag{5.19}$$

$$\Phi'_0 = \frac{\partial S}{\partial \lambda_0} = 0 \tag{5.20}$$

$$H'_0 = \frac{\partial S}{\partial q_0} - \lambda_0 = 0 \tag{5.21}$$

According to Rabei *et al.* (2002) and Nawafleh *et al.* (2004) the general proposed solution for this set of equations can be determined as:

$$S(q_1, q_0, \lambda_0, t) = -E_1 t + W_1(E_1, q_1) + \lambda_0 q_0 + A' \tag{5.22}$$

Substituting Eq. (5.22) in Eq. (5.19), we have

$$-E_1 + \frac{1}{2}\left(\frac{\partial W_1}{\partial q_1}\right)^2 + \frac{1}{2}q_1^2 + \lambda_0 q_1 = 0 \tag{5.23}$$

and this equation leads to

$$W_1(q_1, E_1) = \int \sqrt{2E_1 + \lambda_0^2 - (q_1+\lambda_0)^2}\, dq_1 \tag{5.24}$$

Thus, the Hamilton-Jacobi function becomes

$$S = -E_1 t + \int \sqrt{2E_1 + \lambda_0^2 - (q_1+\lambda_0)^2}\, dq_1 + \lambda_0 q_0 + A' \tag{5.25}$$

The solution for the generalized coordinates are obtained using Eqs. (4.44):

$$\mu_1 = \frac{\partial S}{\partial E_1} = -t + \int \frac{dq_1}{\sqrt{2E_1 + \lambda_0^2 - (q_1+\lambda_0)^2}} \tag{5.26}$$

$$\pi_0 = \frac{\partial S}{\partial \lambda_0} = q_0 - \int \frac{q_1}{\sqrt{2E_1 + \lambda_0^2 - (q_1+\lambda_0)^2}} dq_1 \tag{5.27}$$

These two equations are solved respectively, to give

$$q_1 = \sqrt{2E_1 + \lambda_0^2}\sin(\mu_1 + t) - \lambda_0 \tag{5.28}$$

$$q = -\lambda_1(\mu_1 + t) - \sqrt{2E_1 + \lambda_0^2}\cos(\mu_1 + t) \tag{5.29}$$

This solution is equivalent to the solution obtained in (5.16).

Besides, the generalized momenta can be determined as

$$p_0 = \frac{\partial S}{\partial q_0} = \lambda_0. \tag{5.30}$$

$$p_1 = \frac{\partial S}{\partial q_1} = \sqrt{2E_1 + \lambda_0^2 - (q_1 + \lambda_0)^2} \tag{5.31}$$

Substituting Eq. (5.28) in Eq. (5.31), we have the following solution.

$$p_1 = \sqrt{2E_1 + \lambda_0^2}\cos(\mu_1 + t) \tag{5.32}$$

Again, it is equivalent to solution (5.18).

The wave function for our Lagrangian can be determined using Eq. (4.49) as:

$$\psi(q_1, q_0, \lambda_0, t) = \psi_{01}(q_1)e^{\frac{iS(q_1, q_0, \lambda_0, t)}{\hbar}} \tag{5.33}$$

Where

$$\psi_{01}(q_1) = \frac{1}{\sqrt{p_1(q_1)}} = \left[\left(2E_1 + \lambda_0^2\right) - (q_1 + \lambda_0)^2\right]^{\frac{-1}{4}}.$$

And

$$S = -E_1 t + \int \sqrt{2E_1 + \lambda_0^2 - (q_1 + \lambda_0)^2}\,dq_1 + \lambda_0 q_0 + A'$$

Taking the limit, $\hbar \to 0$, this wave function satisfies the following conditions:

$$\begin{aligned} H_t'\psi &= \left[\frac{\hbar}{i}\frac{\partial}{\partial t} - \frac{\hbar^2}{2}\frac{\partial^2}{\partial q_1^2} + \frac{1}{2}q_1^2 + \lambda_0 q_1\right]\psi = 0; \\ \Phi_0'\psi &= \frac{\hbar}{i}\frac{\partial}{\partial \lambda_0}\psi = 0; \\ H_0'\psi &= \frac{\hbar}{i}\frac{\partial}{\partial q_0}\psi - \lambda_0\psi = 0. \end{aligned} \tag{5.34}$$

6. CONCLUSION

In this paper, the higher order regular Lagrangians treated as first-order singular Lagrangians. In physical terms, this means that each velocity $\frac{dq}{dt}$ replaced with a new function $q_1$and then, the constraint $q_1 - \frac{dq}{dt} = 0$ is added to the original Lagrangian. In the same manner, the acceleration is replaced by q_2 and so on. In other words, the new Lagrangian is of first-order singular Lagrangian.

The extended Lagrangian treated using the Hamilton-Jacobi approach. The action function obtained and the system quantized using the WKB approximation.

Most of the literature in physics on the Hamiltonian treatment of higher order lagrangian used Ostrogradsky's method. This method is written in an ambiguous mathematical language. The time derivatives of the coordinates of different order have to be considered as being independent. In our treatment, the equations of motion are written and the system is quantized using the natural mathematical language. We believe that in this treatment, the local structure of phase space and its local simplistic geometry is made more transparent than in Ostrogradsky's approach.

The generalization of this treatment to N degrees of freedom and to the singular systems is straightforward.

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